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A Note on Essential Self-adjointness of Dirac Operator with a Monopole

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Abstract

The purpose of this paper is to analyse the essential self-adjointness of Dirac operator $H = H_0 + V = c\boldsymbol{\alpha} \cdot (-i\nabla + i\boldsymbol{A}) + \beta m_0 c^2 + V$, where $\boldsymbol{A}$ is the vector potential induced by a monopole. The potential V is assumed to be spherically symmetric and of the form $V = u(r)I_4 + v(r)\beta + iw(r)\beta(\boldsymbol{\alpha} \cdot \boldsymbol{e}_r)$. It is shown that H is essentially self-adjoint under some conditions on the behavior of u, v and w in a neighbourhood of the origin.

Key words. Dirac operator, essential self-adjointness, monopole, complex line bundle, section

§1 Introduction

Since 1976 several authors have investigated the Schrödinger operator with a magnetic field induced by a magnetic monopole (simply called a monopole) [7, 8, 15, 17]. It seems worth-while to throw light upon Dirac operator in such a case [15, 17].

Mathematically, a wave function is described as a section of a vector bundle [3] and a vector potential is represented by a connection form of the principal fibre bundle associated with the vector bundle. In this paper we construct the Hilbert space on which Dirac operator H with a monopole operates and study the essential self-adjointness of H. In the sequel, we use the quantity $q := \frac{eg}{c\hbar}$†) (e: electric charge, g: magnetic charge) as a monopole parameter on the basis of Dirac's quantization condition $2q$ should be an integer [1].

In §2 we build up a line bundle $D^{(q)}$ over $\mathbb{R}^3 \setminus \{0\}$ and another one $E^{(q)}$ over the sphere S^2 with the same structure group $U(1)$. Then we make the Hilbert space $\widetilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ on which H operates and the corresponding one $\widetilde{\Gamma}(S^2, E^{(q)})^4$. Subsequently we define the vector potential $\boldsymbol{A}$ explicitly. Since we assume that the potential V in H is spherically symmetric, we rewrite the unperturbed part H_0 of H so that it may contain radial terms and a generalized spin-orbit coupling operator K (Eq.(2.11)).

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†)c is the speed of light and $\hbar$ is the Planck constant.

In §3, using Wu-Yang's monopole harmonic sections $Y^q_{l,m}(\theta,\varphi)$ [7] which form an orthonormal basis for $\widetilde{\Gamma}(S^2, E^{(q)})$, we decompose $\widetilde{\Gamma}(S^2, E^{(q)})^4$ into the direct sum of the simultaneous eigenspaces $\mathfrak{K}^{(q)}_{j,m,k}$ of J^2, J_3 and K[‡]. The restriction of H to the partial wave subspace $L^2((0,\infty),dr)\widehat{\otimes}\mathfrak{K}^{(q)}_{j,m,k}$, $h_{j,m,k}$, is represented on $L^2((0,\infty),dr)^2$ by radial terms.

In §4 we show under what condition the total Hamiltonian H is essentially self-adjoint on $\Gamma^\infty_0(\mathbb{R}^3\setminus\{0\}, D^{(q)})^4$ (As for $\Gamma^\infty_0(\mathbb{R}^3\setminus\{0\}, D^{(q)})$, see the lower half part of this page.). Arnold-Kalf-Schneider's theorems [16] are useful for the essental self-adjointness of $h_{j,m,k}$. Then we obtain the three main results, Theorems 4.2, 4.3, 4.4 by setting some reasonable assumptions on the behavior of $V = u(r)I_4 + v(r)\beta + iw(r)\beta(\boldsymbol{\alpha}\cdot\boldsymbol{e}_r)$ in a neighbourhood of the origin.

§2 Formulation of Dirac operator with a monopole

We first construct two line bundles. Let $\{W_N, W_S\}$ be an open covering of a base space S^2 as follows:

$$W_N := \left\{(\theta,\varphi);\ 0<\theta<\frac{\pi}{2}+\delta, 0<\varphi<2\pi\right\},\quad \left(0<\delta<\frac{\pi}{2}\right) \tag{2.1}$$

$$W_S := \left\{(\theta,\varphi);\ \frac{\pi}{2}-\delta<\theta<\pi, 0<\varphi<2\pi\right\}. \tag{2.2}$$

A transition function τ_{NS} of $W_N\cap W_S$ into the unitary group $U(1)$ is defined by

$$\tau_{NS}(\theta,\varphi) := e^{2iq\varphi}. \tag{2.3}$$

Using these quantities, we build up a complex line bundle $E^{(q)}$. Subsequently, let $D^{(q)}$ be the pull-back of $E^{(q)}$ by the smooth mapping f of $\mathbb{R}^3\setminus\{0\}$ onto S^2 defined as $f(\boldsymbol{x}) := \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. The open covering $\{\{r;\ r>0\}\times W_N, \{r;\ r>0\}\times W_S\}$ of $\mathbb{R}^3\setminus\{0\}$ is chosen and the transition function $t_{NS}(r,\theta,\varphi)$ of $D^{(q)}$ is essentially the same as that of $E^{(q)}$: $t_{NS}(r,\theta,\varphi) = e^{2iq\varphi}$.

Furthermore, let $\Gamma^\infty_0(\mathbb{R}^3\setminus\{0\}, D^{(q)})$ denote the set of all C^∞-class global sections of $D^{(q)}$ with compact support and $\Gamma^\infty(S^2, E^{(q)})$ the set of all C^∞-class global sections of $E^{(q)}$. They are complex linear spaces. We equip $\Gamma^\infty_0(\mathbb{R}^3\setminus\{0\}, D^{(q)})$ and $\Gamma^\infty(S^2, E^{(q)})$ with an inner product as follows:

$$\langle\eta,\xi\rangle = \int_{\mathbb{R}^3\setminus\{0\}} \eta(r,\theta,\varphi)^*\xi(r,\theta,\varphi)r^2\sin\theta\, drd\theta d\varphi, \tag{2.4}$$

$$\langle\Xi,\Psi\rangle = \int_{S^2} \Xi(\theta,\varphi)^*\Psi(\theta,\varphi)\sin\theta\, d\theta d\varphi. \tag{2.5}$$

[‡] J: total angular momentum operator. See (2.11).

Then we obtain the two Hilbert spaces by completing $\Gamma_0^\infty(\mathbb{R}^3\backslash\{0\}, D^{(q)})$ and $\Gamma^\infty(S^2, E^{(q)})$. We denote them by $\widetilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and $\widetilde{\Gamma}(S^2, E^{(q)})$, respectively.

Obviously we get

$$\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \cong C_0^\infty(0,\infty) \otimes \Gamma^\infty(S^2, E^{(q)}) \tag{2.6}$$

and

$$\widetilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \cong L^2((0,\infty), dr)\widehat{\otimes}\widetilde{\Gamma}(S^2, E^{(q)}). \tag{2.7}$$

Since any wave function satisfying Dirac equation has 4 components, the next decomposition provides a starting point

$$\widetilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \cong L^2((0,\infty), dr)\widehat{\otimes}\widetilde{\Gamma}(S^2, E^{(q)})^4. \tag{2.8}$$

We have now reached the stage of construction of the vector potential in a free Hamiltonian H_0. It must be described with a connection form of the principal fibre bundle associated to $D^{(q)}$. Since the magnetic field induced by a monopole q is a curvature form of the connection form, we choose Wu-Yang's connection form $\mathcal{A}$ [13] and take the vector potential $\boldsymbol{A}$ to be the dual of $\mathcal{A}$:

$$\begin{cases} \boldsymbol{A}_N = \dfrac{iq(1-\cos\theta)}{r\sin\theta}\boldsymbol{e}_\varphi{}^{\S)} & \text{on } \{r;\ r>0\} \times W_N, \\ \boldsymbol{A}_S = \dfrac{-iq(1+\cos\theta)}{r\sin\theta}\boldsymbol{e}_\varphi & \text{on } \{r;\ r>0\} \times W_S. \end{cases} \tag{2.9}$$

With the help of $\boldsymbol{A}$ we can define H_0 as

$$H_0 := c\boldsymbol{\alpha}\cdot(-i\nabla + i\boldsymbol{A}) + \beta m_0 c^{2\P)}. \tag{2.10}$$

We shall here assume that the perturbed potential V is spherically symmetric and that $V(r)$ is 4×4 Hermitian matrix composing of continuous functions on $(0,\infty)$. The total Hamiltonian $H := H_0 + V$ operates on $\widetilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. We take the domain $\mathrm{Dom}(H)$ to be $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ for the present.

To decompose H into the direct sum of radial terms on the basis of (2.8), we rewrite H_0 by four new operators $\boldsymbol{L}, \boldsymbol{S}, \boldsymbol{J}$ and K.

$$\begin{aligned} \boldsymbol{L} &:= \boldsymbol{M} - q\boldsymbol{e}_r, \quad \boldsymbol{S} := \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \\ \boldsymbol{J} &:= \boldsymbol{L}I_4 + \boldsymbol{S}, \quad K := \beta(2\boldsymbol{S}\cdot\boldsymbol{M} + I_4), \end{aligned} \tag{2.11}$$

§) $\boldsymbol{e}_r = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\boldsymbol{e}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$, $\boldsymbol{e}_\phi = (-\sin\varphi, \cos\varphi, 0)$.

¶) α_j $(j = 1,2,3)$, $\beta = \alpha_0$ are 4×4 constant Hermitian matrices satisfying the anti-commutation relations $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_4$.

where $\boldsymbol{M}$ is the auxiliary operator in $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ given by

$$\boldsymbol{M} := \boldsymbol{x} \wedge (-i\nabla + i\boldsymbol{A}). \tag{2.12}$$

Then $\boldsymbol{L}$ is a symmetric operator defined on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and $\boldsymbol{S}, \boldsymbol{J}, K$ are symmetric operators defined on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. The operators $\boldsymbol{J}$ and K are called the total angular momentum operator and the generalized spin-orbit coupling one, respectively. Theses operators enable us to deduce

$$H_0 = -ic(\boldsymbol{\alpha} \cdot \boldsymbol{e}_r)\left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r}\beta K\right) + \beta m_0 c^2. \tag{2.13}$$

§3 Decomposition of Dirac operator

We first decompose $\widetilde{\Gamma}(S^2, E^{(q)})^4$ into the direct sum of simultaneous eigenspaces of $\boldsymbol{J}^2, J_3$ and K. We here put

$$\Xi_q := \left\{|q| - \frac{1}{2}, |q| + \frac{1}{2}, |q| + \frac{3}{2}, \dots\right\} \quad \left(q = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots\right), \tag{3.1}$$

$$\kappa_j^{(q)} := \sqrt{\left(j + \frac{1}{2}\right)^2 - q^2} \quad (j \in \Xi_q). \tag{3.2}$$

There exists an orthonormal basis

$$\{\Phi_{j,m,k}^{\pm} \mid j \in \Xi_q, m = -j, -j+1, \dots, j-1, j, k = \pm\kappa_j^{(q)}\} \tag{3.3}$$

of $\widetilde{\Gamma}(S^2, E^{(q)})^4$ whose elements satisfy the following simultaneous eigenequations of $\boldsymbol{J}^2, J_3$ and K, according to Y. Kazama *et al* [8].

$$\begin{cases} \boldsymbol{J}^2 \Phi_{j,m,k}^{\pm} = j(j+1)\Phi_{j,m,k}^{\pm}, & \\ J_3 \Phi_{j,m,k}^{\pm} = m\Phi_{j,m,k}^{\pm}, & m = -j, -j+1, \dots, j-1, j, \\ K\Phi_{j,m,k}^{\pm} = -k\Phi_{j,m,k}^{\pm}, & k = -\kappa_j^{(q)}, \kappa_j^{(q)}. \end{cases} \tag{3.4}$$

All $\Phi_{j,m,k}^{\pm}$ are constructed with Wu-Yang's monopole harmonic sections $Y_{l,m}^q$ [7].

The above consideration leads us to the following decomposition theorem.

Theorem 3.1. *When setting* $\mathfrak{K}_{j,m,k}^{(q)} := \operatorname{span}\{\Phi_{j,m,k}^{+}, \Phi_{j,m,k}^{-}\}$ *we obtain*

$$\widetilde{\Gamma}(S^2, E^{(q)})^4 \cong \bigoplus_{j \in \Xi_q} \bigoplus_{m=-j}^{j} \bigoplus_{k = \pm\kappa_j^{(q)}} \mathfrak{K}_{j,m,k}^{(q)} \tag{3.5}$$

owing to [7] and [8].

Combination of Eqs.(2.8) and (3.5) yields the relation

$$\widetilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \cong \bigoplus_{j\in\Xi_q} \bigoplus_{m=-j}^{j} \bigoplus_{k=\pm\kappa_j^{(q)}} \left(L^2((0,\infty),dr)\widehat{\otimes}\mathfrak{K}_{j,m,k}^{(q)}\right). \tag{3.6}$$

Each subspace $L^2((0,\infty),dr)\widehat{\otimes}\mathfrak{K}_{j,m,k}^{(q)}$ is called a partial wave subspace and isomorphic to $L^2((0,\infty),dr)^2$.

Assume that V has the form of

$$V(r) = u(r)I_4 + v(r)\beta + iw(r)\beta(\boldsymbol{\alpha}\cdot\boldsymbol{e}_r), \tag{3.7}$$

where u, v and w are real-valued C^1-class functions on $(0,\infty)$. Since $\beta\Phi_{j,m,k}^{\pm} = \pm\Phi_{j,m,k}^{\pm}$ and $-i(\boldsymbol{\alpha}\cdot\boldsymbol{e}_r)\Phi_{j,m,k}^{\pm} = \pm\Phi_{j,m,k}^{\mp}$, we obtain the following fundamental theorem.

Theorem 3.2. *Let $h_{j,m,k}$ denote the restriction of the total Hamiltonian H to the partial wave subspace. Then we have*

$$H \cong \bigoplus_{j\in\Xi_q} \bigoplus_{m=-j}^{j} \bigoplus_{k=\pm\kappa_j^{(q)}} h_{j,m,k} \tag{3.8}$$

and $h_{j,m,k}$ is represented by

$$h_{j,m,k} = \begin{pmatrix} m_0c^2 + u(r) + v(r) & c\left\{-\dfrac{d}{dr} + \dfrac{k}{r}\right\} + w(r) \\ c\left\{\dfrac{d}{dr} + \dfrac{k}{r}\right\} + w(r) & -m_0c^2 + u(r) - v(r) \end{pmatrix} \quad (k = \pm\kappa_j^{(q)}) \tag{3.9}$$

on $C_0^\infty(0,\infty)^2$.

The operator $h_{j,m,k}$ is called a radial Dirac operator.

§4 Essential self-adjointness of Dirac operator

We are now in a position to state a sufficient condition that Dirac operator be essentially self-adjoint. The following theorem serves well for the purpose.

Theorem 4.1. *Let $u, v \in C^1(0,\infty)$ and $f_\pm := u \pm v$. Suppose $\lim_{r\to 0} rf_\pm(r)$ exist. Put $l_\pm := \dfrac{1}{c}\lim_{r\to 0} rf_\pm(r)$. If $l_+l_- < (\kappa_j^{(q)})^2 - \dfrac{1}{4}$, then $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0,\infty)^2$ for all $j \in \Xi_q$.*

The proof is easily given owing to V. Arnold, H. Kalf, and A. Schneider [16].

Theorem 4.2. *Let $g \in C^1(0,\infty)$. If $\lim_{r\to 0} g(r)$ exists and $|g(+0)| > \frac{1}{2}$, then the total Hamiltonian $H = H_0 + \frac{cg(r)}{r}\beta$ is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ for all $|q| \geq \frac{1}{2}$.* $\left(u = w = 0, v = \frac{cg(r)}{r}\right)$

Proof. It is sufficient to prove the essential self-adjointness of the radial Dirac operator $h_{j,m,k}$ for each $j \in \Xi_q$. The constants $\pm m_0c^2$ in the diagonal part of $h_{j,m,k}$ may be omitted in discussion of essential self-adjointness. Then we have

$$-g(+0)^2 < (\kappa_j^{(q)}) - \frac{1}{4}$$

for all $j \in \Xi_q$. Hence $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0,\infty)^2$ for all $j \in \Xi_q$. This implies that H is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. □

Theorem 4.3. *Let $|q| \geq \frac{1}{2}$. If the inequalities $\frac{1}{2} < |b| < \sqrt{2|q|+1} - \frac{1}{2}$ hold, then the total Hamiltonian $H = H_0 + i\frac{cb}{r}\beta(\boldsymbol{\alpha}\cdot\mathbf{e}_r)$ is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$.* $\left(u = v = 0, w = \frac{cb}{r}\right)$

Proof. The constants $\pm m_0c^2$ in the diagonal part may be omitted in argument on the essential self-adjointness of $h_{j,m,k}$.

Case I. $j \geq |q| + \frac{1}{2}$: Assume the inequalities $\frac{1}{2} < b < \sqrt{2|q|+1} - \frac{1}{2}$ hold. In the case of $k = \kappa_j^{(q)}$, we have

$$(b+k)^2 - \frac{1}{4} \geq b^2 - \frac{1}{4} > 0.$$

In the case of $k = -\kappa_j^{(q)}$, we have

$$(b+k)^2 - \frac{1}{4} = \left(b - \kappa_j^{(q)} + \frac{1}{2}\right)\left(b - \kappa_j^{(q)} - \frac{1}{2}\right). \qquad (*)$$

Since $\kappa_j^{(q)} \geq \sqrt{2|q|+1}$, we get $b - \kappa_j^{(q)} \leq -\frac{1}{2}$ and the right-hand side of Eq.(*) is non-negative. Hence it follows from Theorem 4.1 that $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0,\infty)^2$. Likewise in the case of $b < 0$, we can obtain the assertion.

Case II. $j = |q| - \frac{1}{2}$: In this case, we have $0 < b^2 - \frac{1}{4}$ $(\kappa_j^{(q)} = 0)$, and so $h_{j,m,k}$ is essentially self-adjoint.

As a consequence, $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0,\infty)^2$ for all $j \in \Xi_q$. This means that H is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. □

Theorem 4.4. *Let* $|q| \geq \frac{1}{2}$. *Assume that* u *is a* C^1*-class function on* $(0,\infty)$ *and* $p_0 := \frac{1}{c}\lim_{r\to 0} ru'(r)$ *exists. If the inequalities* $\frac{1}{2} < |p_0\lambda| < \sqrt{2|q|+1} - \frac{1}{2}$ *hold, then the total Hamiltonian* $H = H_0 + u(r)I_4 + i\lambda u'(r)\beta(\boldsymbol{\alpha}\cdot \mathbf{e}_r)$ *is essentially self-adjoint on* $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. $(v = 0, w = \lambda u'(r))^{\|)}$

Proof. The constants $\pm m_0c^2$ in the diagonal part may be omitted.

Case I. $j \geq |q| + \frac{1}{2}$: Assume the inequalities $\frac{1}{2} < p_0\lambda < \sqrt{2|q|+1} - \frac{1}{2}$ hold. In a similar way to the proof of Theorem 4.3 we get

$$(k + p_0\lambda)^2 - \frac{1}{4} > 0$$

for $k = \pm\kappa_j^{(q)}$. Likewise in the case of $p_0\lambda < 0$ we obtain the above inequality. Hence it follows from Corollary 2 of Theorem 3 of Ref.[16] that $h_{j,m,k}$ is in the limit-point case at the origin. Consequently, $h_{j,m,k}$ is essentially self-adjoint.

Case II. $j = |q| - \frac{1}{2}$: In this case, we have $0 < (p_0\lambda)^2 - \frac{1}{4}$ $(\kappa_j^{(q)} = 0)$.

The both cases imply that H is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3\setminus\{0\}, D^{(q)})^4$. □

§5 Discussion

In §4 we have proved the essential self-adjointness of H by the limit-point case at the origin of every radial Dirac operator $h_{j,m,k}$ (Theorem 4.1, [16]) and the decomposition theorem (Theorem 3.1 and 3.2). In our case (a monopole exists), it is an interesting fact that although the unperturbed operator

$$h_{j,m,k}^{(0)} = \begin{pmatrix} m_0c^2 & -c\dfrac{d}{dr} \\ c\dfrac{d}{dr} & -m_0c^2 \end{pmatrix}$$

for $j = |q| - 1/2\,(\kappa_j^{(q)} = 0)$ is not essentially self-adjoint, $h_{j,m,k}$ becomes essentially self-adjoint if H has a special-type potential.

The investigation of the essential self-adjointness the usual n-dimensional Dirac operator was treated by Kalf and Yamada [19]. Under the assumption that m and V are spherically symmetric, they reduced the problem to that of every radial Dirac operator h with $k \in \pm\{\mathbb{N}_0 + (n-1)/2\}$. Their method$^{**)}$ is the same as ours. But since $k = \pm\sqrt{(j+1/2)^2 - q^2}$ and $j \in \Xi_q$ in our case, it is more difficult to study the essential self-adjointness of $h_{j,m,k}$.

$^{\|)}$Behncke and Thaller already discussed this case for the usual Dirac operator (No monopole) [10, 14]. *cf.* Corollaries 2 and 3 of Theorem 3 in [16].

$^{**)}$Kalf and Yamada's varying mass term $m(r)$ corresponds to $m_0c^2 + v(r)$ in our case.

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